Strength Criteria for Anisotropic Brittle Materials

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The criterion of strength for anisotropic media proposed by Gol'denblat and Kopnov is adopted in this paper. The invariants having transversely isotropic material symmetry are used to formulate a rational criterion of strength with the effects of hydrostatic stress and the Baushinger. For illustration, a quadratic strength function for the media is formulated explicitly in terms of engineering strengths; also, a strength surface for the AGOT graphite, which exhibits transversely isotropic and brittle properties, is presented as a numerical example.

Introduction

FRACTURE in solids is initiated by some flaw or imperfection, such as a microcrack or inclusion. The flaw or the imperfection causes a high stress concentration in that region. For a sufficiently high value of the local stress, the bonds between atoms at the microcrack edge may be weakened, and dislocations between layers of atoms near the edge of the microcrack may occur. As the process is continued, the microflaws in a solid grow into sizable fracture surfaces and link up together, finally causing complete fracture of the solid. However, the analytical treatment of the strength of atomic bonds in solids is cumbersome and it gives only the necessary, but not the sufficient, condition of the final fracture. Hence, an alternative, continuum mechanics approach to the problem of stength of materials will be adopted in this paper. Generally, fracture in solids may be categorized as brittle or ductile. With brittle fracture, there is little if any plastic deformation or shape distortion, but the volume change may be large. A ductile fracture, however, is preceded by considerable plastic deformation or yielding before the actual failure occurs. Most fractures in solids involve both aforementioned modes of fracture, but are usually dominated by one. In a crystallographic sense, brittle fractures generally occur by cleavage, where the tensile stresses literally pull apart adjacent planes of atoms; ductile fractures occur after a sequence of shear deformation that causes the planes of atoms to slip with respect to each other. The sequence of shear deformation changes the material properties continuously in a complex way. Therefore, the ductile fracture cannot be predicted simply. In the study of strength of materials, the ductile fracture and the initial and subsequent yieldings are considered separately.

Because a rational and simple strength criterion for brittle materials is essential and is important for the purpose of material characterization and design, the criterion of strength for anisotropic brittle materials is investigated in this paper. For simplicity, the effects of time, temperature, strain-gradient, size, and microstructural atoms of solids are neglected. The failure states of stress for anisotropic brittle solids are sensitive to the mean normal stress, which has no effect on the initial yielding surface of a ductile material.

As pointed out by Tsai and Wu,[1] the majority of the proposed criteria are limited in their usage because they do not include the correlating stress effects. In order to remove such a limitation, Gol'denblat and Kopnov[2] proposed a new criterion of strength for glass-reinforced plastics that behave as an orthotropic material. They verified their results experimentally and showed the suitability of the proposed criterion of strength for practical usage.

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Basic Consideration

General Theory

The general theory of strength functions for anisotropic materials can be established[3] from consideration of the strength function F,

$$F(\sigma_{ij})=0 \tag{1}$$

where σ_{ij} is the stress tensor, which is, of course, symmetric. The strength function F is required to be invariant under the group of transformation $\{t_{ij}\}$ that characterizes the material anisotropy,

$$F(\bar{\sigma}_{ij})=F(\sigma_{ij}) \tag{2}$$

where the transformed stress tensor obeys the rule

$$\bar{\sigma}_{ij}=t_{ir}t_{js}\sigma_{rs}$$

It is also assumed that the strength function F can be expressed approximately in the following form as proposed by Gol'denblat and Kopnov[2]:

$$F=(F_{ij}\sigma_{ij})^{\alpha}+(F_{ijkl}\sigma_{ij}\sigma_{kl})^{\beta}+(F_{ijklmn}\sigma_{ij}\sigma_{kl}\sigma_{mn})^{\gamma}-1 \tag{3}$$

which, in fact, is a third-order polynomial approximation of Eq. (1). In Eq. (3), the F_{ij}, F_{ijkl}, and F_{ijklmn} are strength tensors of rank two, four, and six, respectively. The power α, β, and γ are material constants. For the case of $\alpha=\beta=\gamma=1$, the tensors F_{ij} and F_{ijklmn} characterize the Bauschinger effect of materials and the tensors F_{ijkl} and F_{ijklmn} determine the hypersurface of the strength function in the six-dimensional stress-space.

Strength Functions for Anisotropic Materials

In this section, we list for each of the following crystal classes a set of quantities $\{I_1^{(1)},\ldots,I_n^{(3)}\}$, each of which is a polynomial of degree three or lower in the six stress components $(\sigma_{11},\sigma_{22},\ldots,\sigma_{12})$ and is invariant under the group of transformations $\{t_{ij}\}$ associated with the given crystal class. Hence, any polynomial of degree three or lower in $\sigma_{11},\ldots,\sigma_{12}$, which is invariant under $\{t_{ij}\}$, is expressible as a combination of $I_1^{(1)},\ldots,I_n^{(3)}$. It should be noted that only terms of degree three or lower are of interest. Those terms of degree four and higher are omitted in the following results. The set of I_i for each of the crystal classes has been determined by the author.[4,5] In this paper, only the triclinic, rhombic (orthotropic), and transversely isotropic systems are listed. Also, in the following results, the invariants I_i are given with contracted notation, i.e.,

$$(\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{23},\sigma_{13},\sigma_{12}) \equiv (\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6)$$

1) Triclinic system.

Pedial **I** $\{t_{ij}\}$

Pinacoidal **I,C**

For crystals having triclinic symmetry, there is no restriction on the orientation of the preferred direction; any rectangular coordinate system can be used as a reference frame. Thus, the invariants $\{I_j\}$ for both classes are

$$I_i^{(1)}: \quad \sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6 \tag{4}$$

2) Rhombic system.

Rhombic-pyramidal **I,R$_2$,R$_3$,D$_1$**

Rhombic-disphenoidal **I,D$_1$,D$_2$,D$_3$** $\{t_{ij}\}$

Rhombic-dipyramidal **I,C,R$_1$,R$_2$,R$_3$,D$_1$,D$_2$,D$_3$**

$$I_i^{(1)}: \quad \sigma_1,\sigma_2,\sigma_3$$

$$I_i^{(2)}: \quad \sigma_4^2,\sigma_5^2,\sigma_6^2 \qquad \{I_i\}$$

$$I_i^{(3)}: \quad \sigma_4\sigma_5\sigma_6 \tag{5}$$

3) Transverse isotrophy. It is supposed that the material is transversely isotropic with respect to the axis x_3. Thus, the transformations characterizing transverse isotropy are **I** and $\mathbf{R}_\alpha\{x_1'+ix_2'=e^{-i\alpha}(x_1+ix_2),\ x_3'=x_3 \mid$ for all values of $\alpha\}$. Therefore, the invariants are

$$I_i^{(1)}: \quad \sigma_3,\sigma_1+\sigma_2$$

$$I_i^{(2)}: \quad \sigma_1\sigma_2-\sigma_6^2,\ \sigma_4^2+\sigma_5^2 \qquad \{I_i\}$$

$$I_i^{(3)}: \quad \{\det|\sigma_i|\} \tag{6}$$

When the terms of degree three $\{I_i^{(3)}\}$ for F are omitted, the resulting equations (4-6) yield the forms of the quadratic strength function. In particular, the strength function for an orthotropic material can be written readily from Eq. (5) with $\{I_i^{(3)}\}$ omitted. That is,

$$(F_1\sigma_1+F_2\sigma_2+F_3\sigma_3)^\alpha+(F_{11}\sigma_1^2+F_{22}\sigma_2^2+F_{33}\sigma_3^2+2F_{12}\sigma_1\sigma_2$$
$$+2F_{23}\sigma_2\sigma_3+2F_{13}\sigma_1\sigma_3+F_{44}\sigma_4^2+F_{55}\sigma_5^2+F_{66}\sigma_6^2)^\beta=1 \tag{7}$$

This result can be reduced to the results in Ref. 1 by taking $\alpha=\beta=1$ and in Ref. 2 by taking $\alpha=1$ and $\beta=½$. Furthermore, if the absence of a Bauschinger effect is assumed, the material constants α and β are taken to be unity and the incompressibility of materials during yielding is imposed, Eq. (7) yields the Hill criterion,[6]

$$F(\sigma_2-\sigma_3)^2+G(\sigma_3-\sigma_1)^2+H(\sigma_1-\sigma_2)^2$$
$$+2L\sigma_4^2+2M\sigma_5^2+2N\sigma_6^2=1 \tag{8}$$

with the following symbols defined by Hill[6]:

$$F=-F_{23},\quad G=-F_{13},\quad H=-F_{12},\quad 2L=F_{44}$$
$$2M=F_{55},\quad 2N=F_{66}$$

and the following conditions of incompressibility:

$$F_{11}=G+H,\quad F_{22}=H+F,\quad \text{and} \quad F_{33}=F+G$$

For a transverse isotropic material with axis x_3 as the symmetric axis, the quadratic strength function F can be obtained directly from Eq. (6),

$$F_1(\sigma_1+\sigma_1)+F_3\sigma_3+F_{11}(\sigma_1^2+\sigma_2^2)+F_{33}\sigma_3^2+F_{44}(\sigma_4^2+\sigma_5^2)$$
$$+2(F_{11}-F_{12})\sigma_6^2+2F_{12}\sigma_1\sigma_2+2F_{13}[(\sigma_1+\sigma_2)\sigma_3]=1 \tag{9}$$

This result has been given by Tsai and Wu.[1]

Determination of Components of the Strength Tensors

In this section, the representation of the strength surface in six-dimensional stress-space is determined. The axes of elastic symmetry of orthotropy are used as coordinate axes (x_1,x_2,x_3). By taking $\alpha=\beta=1$ in Eq. (7), the quadratic form of the strength function can be expressed as the matrix form,

$$F_i\sigma_i+F_{ij}\sigma_i\sigma_j=1 \tag{10}$$

in which

$$F_i=\begin{vmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \\ 0 \\ 0 \end{vmatrix} \quad \text{and} \quad F_{ij}=\begin{vmatrix} F_{11} & F_{12} & F_{13} & 0 & 0 & 0 \\ F_{12} & F_{22} & F_{23} & 0 & 0 & 0 \\ F_{13} & F_{23} & F_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & F_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & F_{66} \end{vmatrix}$$

The 12 independent strength components $F_1,\ldots,F_{66}$ are intrinsic material parameters and can be determined experimentally from the basic engineering strengths of an orthotropic medium as follows.

The measures of uniaxial tensile strengths in the directions of the (x_1,x_2,x_3) axes are

$$X_t, Y_t, Z_t$$

respectively. The measures of uniaxial compressive strengths in the (x_1,x_2,x_3) axes are

$$X_C, Y_C, Z_C$$

respectively.

For anisotropic materials, it is important to note that the shearing strength may or may not depend upon the direction of the shear stresses. In particular, for the fiber-reinforced composites, the shearing strength depends upon the directions of the axes of material symmetry.[7] The shearing stresses at failure in the planes of symmetry $(x_1x_2,\ x_1x_3,\ x_2x_3)$ along the new directions x_i' and x_j' $(i,j=1,2,3)$, which make an angle of 45 deg from the material axes x_i and x_j (1,2,3), are

$$(S_t^{(45°)},S_c^{(45°)}),\ (R_t^{(45°)},R_c^{(45°)}),\ \text{and}\ (Q_t^{(45°)},Q_c^{(45°)})$$

respectively.

Table 1 Strengths of AGOT graphite

	psi	MPa		psi	MPa
X_c	5849	40.34	Z_t	1900	13.10
X_t	1400	9.66	$R_c^{(45°)}$	2450	16.90
Z_c	5700	39.31	$R_t^{(45°)}$	1740	12.00

Utilizing Eq. (10), the following results for components of the strength tensors can be readily verified:

$$F_1=\frac{1}{X_t}-\frac{1}{X_c},\qquad F_2=\frac{1}{Y_t}-\frac{1}{Y_c},\qquad F_3=\frac{1}{Z_t}-\frac{1}{Z_c}$$

$$F_{11}=\frac{1}{X_tX_c},\qquad F_{22}=\frac{1}{Y_tY_c},\qquad F_{33}=\frac{1}{Z_tZ_c}$$

$$F_{13}=-\frac{1}{2}\left(\frac{1}{R_t^{(45°)}R_c^{(45°)}}-\frac{1}{Z_tZ_c}-\frac{1}{X_tX_c}\right)$$

$$F_{23}=-\frac{1}{2}\left(\frac{1}{Q_t^{(45°)}Q_c^{(45°)}}-\frac{1}{Y_tY_c}-\frac{1}{Z_tZ_c}\right)$$

$$F_{12}=-\frac{1}{2}\left(\frac{1}{S_t^{(45°)}S_c^{(45°)}}-\frac{1}{X_tX_c}-\frac{1}{Y_tY_c}\right)$$

$$F_{44}=\frac{1}{Q_t^{(45°)}Q_c^{(45°)}},\qquad F_{55}=\frac{1}{R_t^{(45°)}R_c^{(45°)}}$$

$$F_{66}=\frac{1}{S_t^{(45°)}S_c^{(45°)}} \tag{11}$$

The seven strength components $F_1,\ldots,F_{44}$ [see Eq. (9)] for a transversely isotropic material with the x_3 axis as the isotropic axis are given as follows:

$$F_1=\frac{1}{X_t}-\frac{1}{X_c},\qquad F_3=\frac{1}{Z_t}-\frac{1}{Z_c},\qquad F_{11}=\frac{1}{X_tX_c}$$

$$F_{33}=\frac{1}{Z_tZ_c},\qquad F_{12}=-\frac{1}{2}\left(\frac{1}{S^2}-\frac{2}{X_tX_c}\right)$$

$$F_{13}=\frac{1}{2}\left(\frac{1}{R_t^{(45°)}R_c^{(45°)}}-\frac{1}{X_tX_c}-\frac{1}{Z_tZ_c}\right)$$

$$F_{44}=\frac{1}{Q_t^{(45°)}Q_c^{(45°)}} \tag{12}$$

where S denotes the shearing strength in the isotropic plane x_1x_2, and

$$S=S_t^{(45°)}=S_c^{(45°)}$$

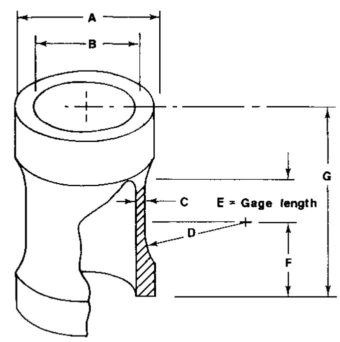
Experimental and Numerical Examples

In order to verify the proposed criterion, the experimental results for grade AGOT graphite are used. This material has the transversely isotropic symmetry. The isotropic axis is chosen along the x_3 axis. The tests of strength for the grade AGOT graphite were carried out by Hackerott and Garibay at Kansas State University.[8,9] The AGOT graphite used for the experimental tests is a fine-grain material with high porosity. It has a purity of 99.6% carbon with a maximum grain size of 1/32 in. and a density of 1.72 g/cm^3. Like most extruded graphites, AGOT is considered to have a preferred crystal orientation with respect to the extruded axis (x_3 axis). Thus, its mechanical properties are transversely isotropic and brittle. The material exhibits both the effect of hydrostatic stress and the Bauschinger effect on fracture.

In most of the previous works on biaxial strength tests of AGOT graphite, the test cylinders have had a small diameter and a correspondingly thin wall.[10,11] In Refs. 8 and 9, the test cylinders were designed with a relatively large diameter and a relatively thick wall to reduce the chance of local buckling. The transition from the end section to the gage section was designed to minimize stress concentrations and to assure that fracture takes place in the gage section. The geometrics of the test cylinders are given in Fig. 1. Using these cylinders, data were generated for compression-compression, compression-tension, and tension-compression cases. The friction between the platens and the end sections of the pressured cylinder was evaluated by cycling the axial load. The loading and unloading stress-strain curves of the graphite followed different paths and show the hysteresis loops, but they were repeatable and returned to the origin each time. This fact indicates that the friction between the platens and the specimen is negligible. The test cylinders were bonded to the platens with approximately ⅛ in. thick adhesive coating of Sikadur Hi-Mod Gel-390, which has a 3000 psi tensile strength after cure.

The AGOT graphite has a relatively high porosity of 20%. Barrier coatings were applied to prevent the pressurizing fluid from penetrating into the test samples. An easily applied, inexpensive paraffin wax barrier was selected.

Uniaxial compression specimens 0.5 in. in diameter and either 2 or 2.5 in. long were used for the compression tests. The "dog-bone" shape tensile specimens of 0.5 in. diameter



	A	B	C	D	E	F	G	E/C	(A+B)/2C
LARGE CYLINDER	3.75	2.75	0.25	2.50	1.00	2.00	5.00	4.00	6.50
	3.75	2.75	0.20	3.75	0.50	2.25	5.00	2.50	8.13
SMALL CYLINDER	2.00	1.50	0.125	2.50	1.00	1.25	3.50	8.00	14.00

Fig. 1 Test specimen geometries (in inches).

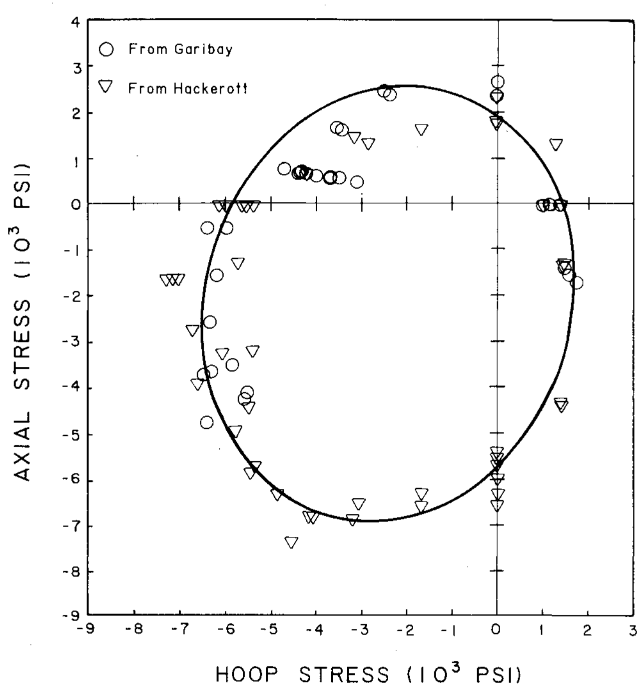


Fig. 2 Strength surface for AGOT graphite.

Table 2 Values of F_i for AGOT

	F_1	F_3	F_{11}	F_{33}	F_{13}
psi, $\times 10^{-4}$	5.43	3.509	1.221[a]	9.234	−1.006
Pa	3.747	2.420	0.842	6.368	−0.694

[a] $\times 10^{-3}$.

and 1.5 in. gage length were used for uniaxial tensile tests, which were conducted by using bonded joints and a cable pull arrangement.

The details of the test apparatus, specimen preparation and seating arrangement, and experimental techniques are described in Refs. 8 and 9. The engineering strengths of the grade AGOT graphite are summarized in Table 1.

The strength function for the material can be derived from Eq. (9) and written as

$$(F_{1\,1}+F_{3\,3})+(F_{11\,1}^{\,2}+2F_{13\,1\,3}+F_{33\,3}^{\,2})=1 \qquad (13)$$

where F_i and F_{ij} for AGOT can be calculated. Using Eq. (13), the F_i and F_{ij} are calculated as shown in Table 2.

The proposed strength function for grade AGOT graphite is shown in Fig. 2. The experimental data points are also plotted for comparison.

Conclusions

In this study of the strength of materials, the difficulties encountered are associated with heterogeneity and with the time and temperature dependencies of the material. Thus, for simplicity, the assumption is made herein that the material is homogeneous and time and temperature independent. Also, the strength of the material is assumed to depend on the "mean" square of the shear stress on all slipping planes. Hence, the concept of strength tensors, which has been introduced by Gol'denblat and Kopnov, can be used for the determination of a criterion governing the stress of an anisotropic brittle material. From the invariants of the stress tensor, a strength function in the quadratic form can be established. The polynomial coefficients (the components of the strength tensors) are determined experimentally from the intrinsic strength data. From Fig. 2, it is seen that the agreement between the proposed criterion and the experimental results is very good except in the second quadrant. It is believed that the higher degree of invariants, such as the third-degree invariants, should be included in the proposed strength function. Furthermore, this criterion differs from earlier criteria, such as the Coulomb-Mohr and Griffith, in that the Bauschinger effect, compressibility, and stress interactions are taken into consideration.

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